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LETTER TO THE EDITOR

Critical probabilities for diversity and number of clusters in randomly occupied square lattices

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Abstract. We report the occurrence of critical probabilities associated with the maximum of diversity and the maximum number of fragments (clusters) on a two-dimensional square lattice. Some scaling relations of these two variables are observed in accordance with work on fragmentation processes.

Recently, the diversity of mass of fragments has been proposed as a measurement of the complexity in aggregation and fragmentation processes [1]. The concept of diversity appears in various problems in biology [2], evolution [3], self-organization and cellular automata [4, 5], fractals [6, 7] and non-equilibrium phenomena [8]. In the last few years the diversity of size or mass has been studied from the point of view of computer simulations in several dissipative processes and cellular automata which generate a distribution of clusters and are of interest in physics, chemistry, biology and ecology [9–11]. In this letter we show that in the problem of the random occupation of a square lattice with probability p, two critical probabilities, $P_c(N_{max})$ and $P_c(D_{max})$, respectively, appear in connection with two statistical variables, namely the total number of fragments

$$N(p) = \left\langle \sum_s N(s, p) \right\rangle \tag{1}$$

and the diversity of mass of fragments,

$$D(p) = \left\langle \sum_s \Theta[N(s, p)] \right\rangle. \tag{2}$$

In these expressions, $N(s, p)$ is the number of fragments of size s, in a single experiment, for occupation probability p; $\Theta(x) = 1$ if $x > 0$ and $\Theta(x) = 0$ otherwise and the averaging $\langle\,\rangle$ is over different experiments. A fragment is defined here as a collection of occupied sites connected by nearest-neighbour relationships.

We perform Monte Carlo simulations on square lattices with size varying from $L^2 = 32^2$, 64^2, 128^2, 256^2, 512^2, 1024^2 and 2048^2, with averages taken on 2000, 1000, 800, 800, 400, 200 and 50 experiments. The lattices were randomly occupied with probability p ranging from 0.05 to 0.95 with steps of 0.01 between 0.15 to 0.60 and steps of 0.05 on the other regions. The variables N and D were measured as functions of both L and p.

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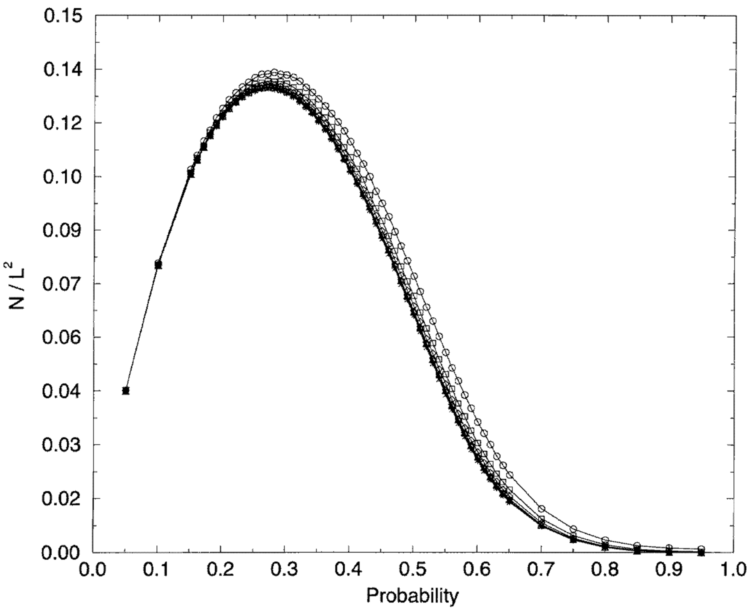

Figure 1. Collapsing curves N/L^2 against the probability of occupation p, obtained from seven different values of L ranging from 32 to 2048.

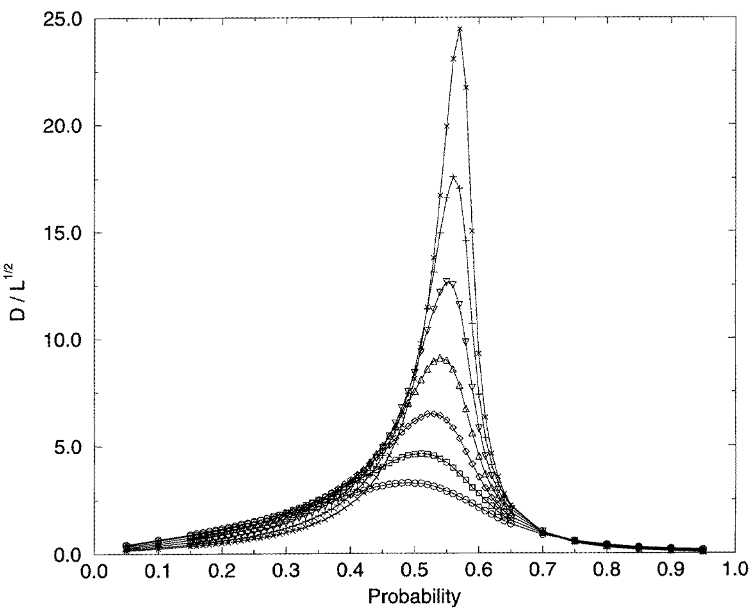

Figure 2. The ratio $D/L^{1/2}$ as a function of p for $L = 2048$ ($\times$), 1024 ($+$), 512 ($\triangledown$), 256 ($\triangle$), 128 ($\Diamond$), 64 ($\Box$), 32 ($\bigcirc$).

In figure 1 we have a plot of N normalized by L^2, so that all the curves collapse independent of the lattice size. N increases with p until it attains a configuration where the

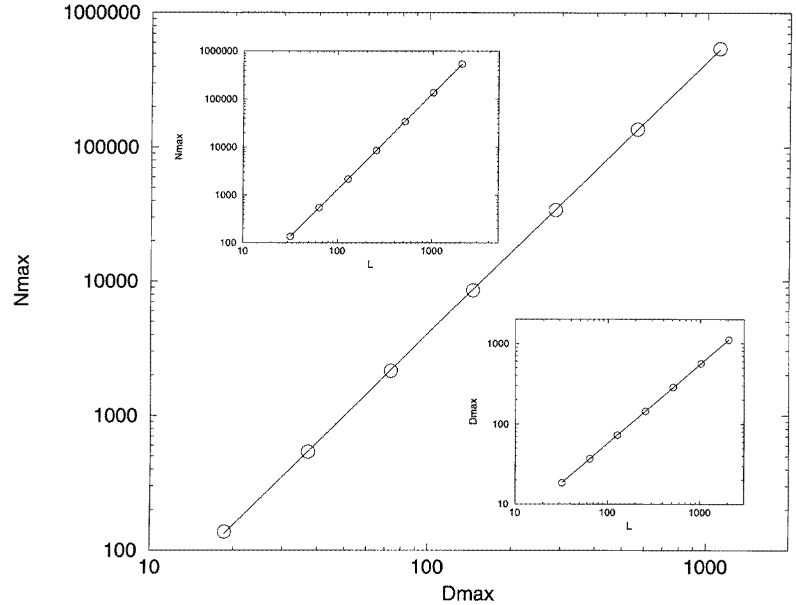

Figure 3. Log–log plot of $N_{\max}$ versus $D_{\max}$. The straight line has a slope of value 2, giving the robust scaling relation $N_{\max} \sim D_{\max}^2$. The insets shows the log–log plots of $D_{\max}$ versus L and $N_{\max}$ against L.

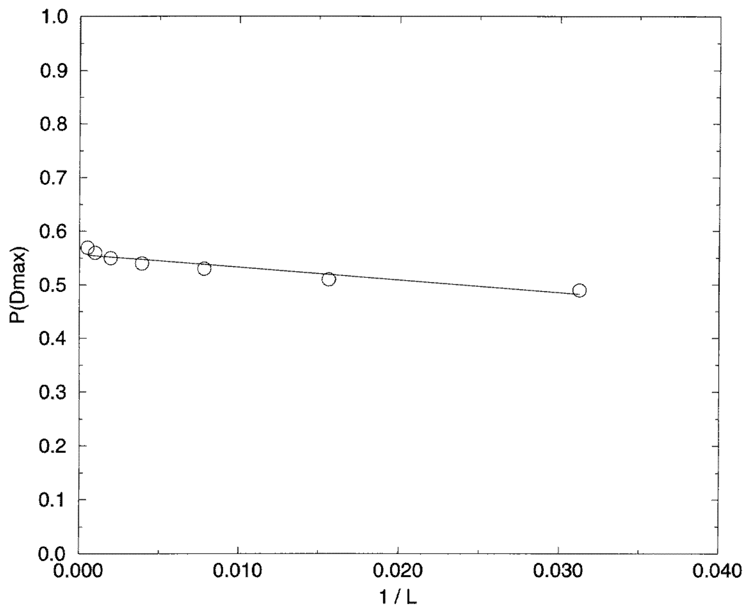

Figure 4. Plot of the linear fit for $P(D_{\max})$ as a function of $1/L$, the straight line has an intercept at 0.55 ± 0.02.

maximum, $N_{\max}$, is reached and then decreases afterwards. The shape of this plot is the same for all the different values of L and the maximum is obtained in a fixed probability.

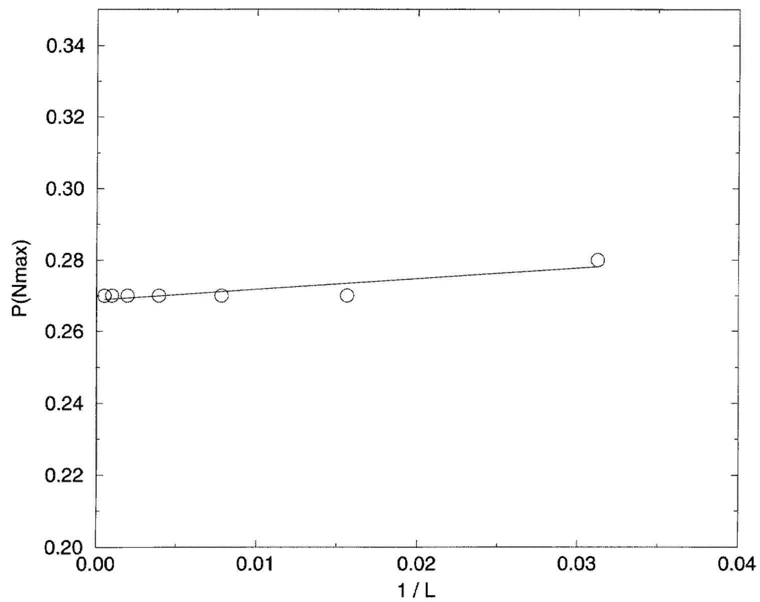

Figure 5. Similar to graph in figure 4 but for $P(N_{\mathrm{max}})$ as a function of $1/L$, giving $P_{\mathrm{c}}(N_{\mathrm{max}}) = 0.27 \pm 0.01$.

The diversity density D/L^2 written as a function of p does not collapse the curves for all Ls. It is necessary to include another scaling factor which we assume to be $D/L^{2+\Delta}$ and Δ is in principle some function of p added to the exponent of L. In figure 2 is shown the dependence of $D/L^{2+\Delta}$ on p. For the best collapsing curve we found that $2+\Delta = 1/2$ and so $\Delta = -3/2$. However the curves do not collapse well in the interval $0.45 \leqslant p \leqslant 0.65$, showing that diversity is rather complex and Δ cannot be simply a constant. In this graph two characteristics must be noted, first that all curves present a definite probability where the maximum of diversity is attained. Second, the shape of the curves changes with L. Doing a kurtosis analysis of the degree of peakedness we found that for smaller L the curves present a mesokurtic distribution and as L increases it changes to a leptokurtic distribution. This indicates that the rate of increase in diversity for different Ls is higher in the maximum region. So, for a system with larger size we expect to find a smaller region in p where a high diversity or complexity occur, showing that diversity can be tuned by parameters p and L.

In these simulations we also found the robust scaling relation $N_{\mathrm{max}} \sim D^2_{\mathrm{max}}$ shown in figure 3. This scaling was observed in different fragmentation and aggregation dynamics on lattices of various dimensionalities [1, 8, 11]. In the insets we have the scaling relations $D_{\mathrm{max}} \sim L$ and $N_{\mathrm{max}} \sim L^2$ observed in our simulations.

To determine the critical probabilities associated with D_{max} and N_{max} at the thermodynamic limit, that is for $L \to \infty$, we plot $P(D_{\mathrm{max}})$ versus $1/L$ as shown in figure 4. A linear fit gives us $P_{\mathrm{c}}(D_{\mathrm{max}})$ equals 0.55 ± 0.02. Note that $P_{\mathrm{c}}(D_{\mathrm{max}})$ is smaller than the percolation threshold for the site percolation problem on a two-dimensional square lattice. That is so because the percolation cluster spanning the lattice does not make room for the emergence of clusters of intermediary sizes.

In figure 5 we have the plot of $P(N_{\mathrm{max}})$ versus $1/L$ giving $P_{\mathrm{c}}(N_{\mathrm{max}}) = 0.27 \pm 0.01$. For large N we need a configuration where smaller and isolated clusters occur, so it is reasonable that $P_{\mathrm{c}}(N_{\mathrm{max}})$ has a lower value than $P_{\mathrm{c}}(D_{\mathrm{max}})$. In an ideal situation N would attain its maximum on a configuration in which an occupied site is followed by an empty site throughout a line. The next line would be a shift of the first and so on. Note that for this configuration one would say that p should have the value of 0.5. But taking into consideration the neighbourhood relations, in this case the four nearest neighbours, and the unlikelihood that this ideal situation could happen, the overall value of p for N to attain its maximum is lower than 0.5 as is shown in figure 1.

In conclusion, we have described the behaviour of cluster size diversity, D, and number of fragments (clusters), N, in a randomly-occupied square lattice with probability p. An interesting tuning effect for N and D has been shown and two new critical probabilities were introduced in this problem related to the percolation system [12]. In the light of recent results [1], the occurrence of $P_{\mathrm{c}}(D_{\mathrm{max}})$ is significant since it is a critical probability associated with the maximum complexity of the system.

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